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RESEARCH MEMORANDUM

SOME EXPERIMENTS ON THE TRAVELING-SALESMAN PROBLEM

by

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SUMMARY

This paper presents the results of a series of experiments on the traveling-salesman problem. The purpose of these experiments was to investigate the efficiency of the linear-programming technique in solving this type of problem.

SOME EXPERIMENTS ON THE TRAVELING-SALESMAN PROBLEM

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1. INTRODUCTION AND DISCUSSION OF RESULTS

During the early part of 1954, Dantzig, Fulkerson, and Johnson formulated a linear-programming technique for the solution of the traveling-salesman problem [1]. The methods employed by these authors quickly generated optimal tours when a good tour was known. The question naturally arose concerning the efficiency of the technique when a random tour was chosen as the starting point. Dr. I. Heller conjectured that it might be efficient and proposed that experiments be made as one means to investigate this question and to gain a greater degree of insight into the dynamics of the technique itself.

A series of 10 nine-city traveling-salesman problems was considered. In each case the distances between cities were chosen from a table of two-digit random numbers. The starting tour was then chosen to be in the order of the natural numbers (with the cities numbered 1 through 9 at the outset). In the last problem, the technique was applied first to find the maximal tour and then to proceed from it to the minimal tour.

Tables 1 to 10 are the distance tables for the 10 problems. An element $a_{ij}(a_{ij} = a_{ji})$ in the distance table represents the distance from the i-th city to the j-th city. It is to be noted that since these distances were chosen from a table of random numbers, they cannot be construed necessarily to represent straight-line distances in Euclidean space. At the foot of each distance table is a table listing the tours obtained at the end of each iteration* (the last entry being the optimal tour). Table 11 lists the tours obtained by starting with the maximal tour associated with Table 10 and proceeding to the minimal tour.

The largest number of iterations needed was six, while the average was only a little under four. The average time to work one of the problems was about 3 hours. Since for a nine-city problem there are $\frac{8!}{2} = 20{,}160$ possible tours, it is apparent that the simplex method which was employed was extremely efficient. In addition, it is noteworthy that the only secondary constraints introduced in the solution of these problems were upper bounds.

In connection with these experiments, A. W. Boldyreff suggested an approximation procedure, the merit of which

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*One iteration is defined to be the process of going from one tour to another tour of the same or shorter length.

lies in its inherent simplicity and in the rapidity with which it may be applied. An application of this approximation method to the 49-city problem of [1] gave a tour of 851 units as compared with the optimal of 699 units, an error of 21%.

2. BRIEF REVIEW OF THE TECHNIQUE OF SOLUTION

The mathematical technique found in [1] will briefly be reviewed here, and will then be illustrated by way of an example.

The factors π_i, referred to as "potentials," are computed from the formula

$$(1) \qquad \pi_i + \pi_j - a_{ij} = 0 \qquad \text{for all } (i,j) \text{ in the basis,}$$

i.e., for all (i,j) corresponding to column vectors in the basis.

Computationally, this means first finding a loop with an odd number of links which correspond to vectors in the basis; then starting at a city k of the loop, adding and subtracting the lengths of the links alternately; the resulting number is then $2\pi_k$. Having π_k, we can then compute all the π_i from (1).

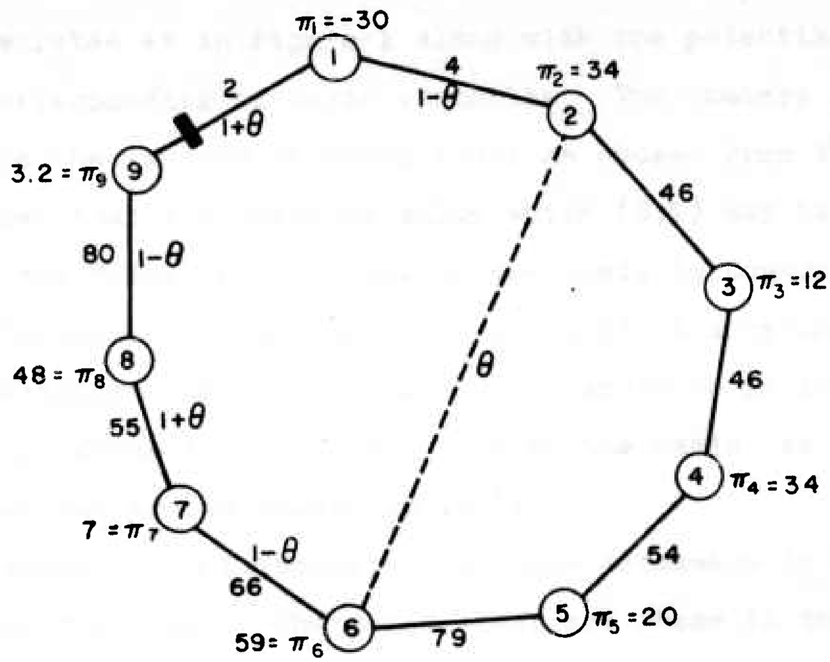

Figure 1

We next choose the link (k,ℓ) such that

$$(2) \qquad \pi_k + \pi_\ell - a_{k\ell} = \max\,(\pi_i + \pi_j - a_{ij}) > 0$$

and allow the value of the corresponding variable $x_{k\ell}$ to be equal to 0. The values of the basic variables must now be corrected so that the sum of the values of link variables around any city must equal 2. For this purpose we try to find an even loop containing (k,ℓ). For example, suppose the starting tour

is illustrated as in Figure 1 along with the potentials and links corresponding to basic variables. The numbers on the links are the lengths of those links as chosen from Table 1. It is seen that the greatest value which (6,2) may have is 0 since the value of each link in the basis is bounded above by 1. Consequently, we put a bar on (9,1), indicating that it has attained its upper bound and is replaced in the basis by (6,2). Since (9,1) is no longer in the basis, we must recompute the prices according to (1).

Figures 1 – 5 illustrate the steps necessary to complete the first iteration. The solid lines are those in the tour. In Figure 5 a new tour is achieved. By repeating the process we ultimately reach the optimal tour when

(3) $$\pi_i + \pi_j - a_{ij} \leq 0 \quad \text{for all } i,j = 1,2,\ldots,9.$$

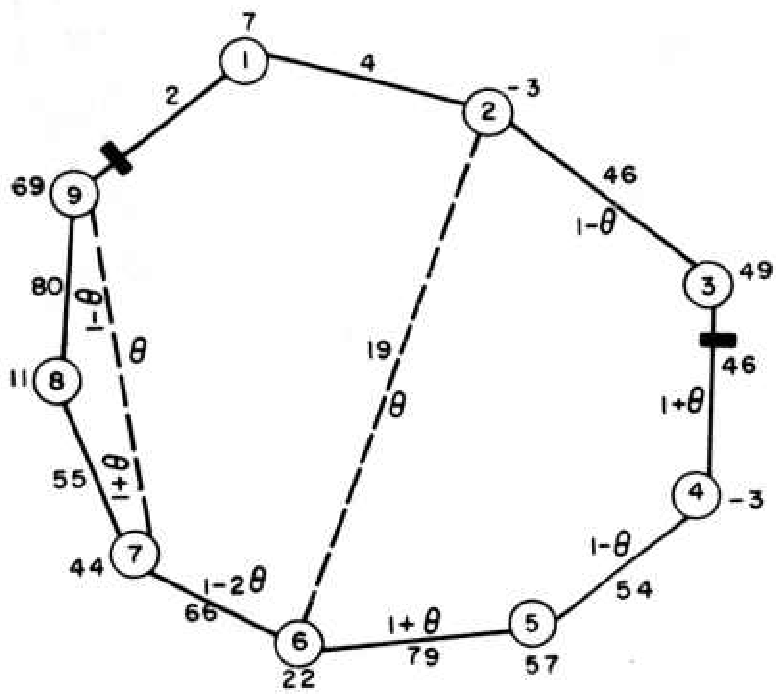
7
4
2
-3
46
69
1-θ
49
80
1-θ
θ
19
11
46
θ
1+θ
55
1+θ
-3
44
1-2θ
1-θ
66
1+θ
54
79
57
22

Figure 2

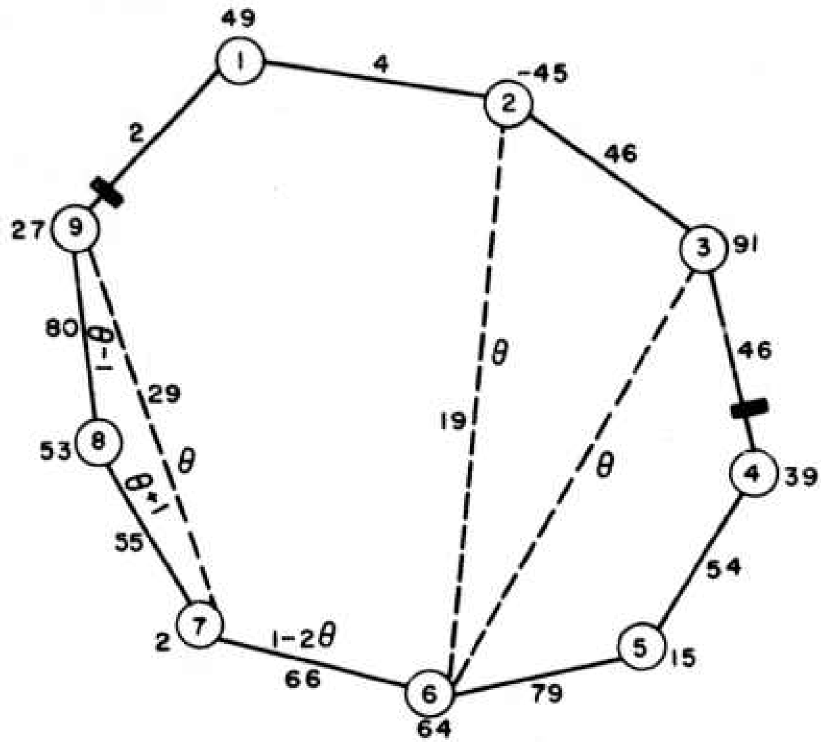

Figure 3

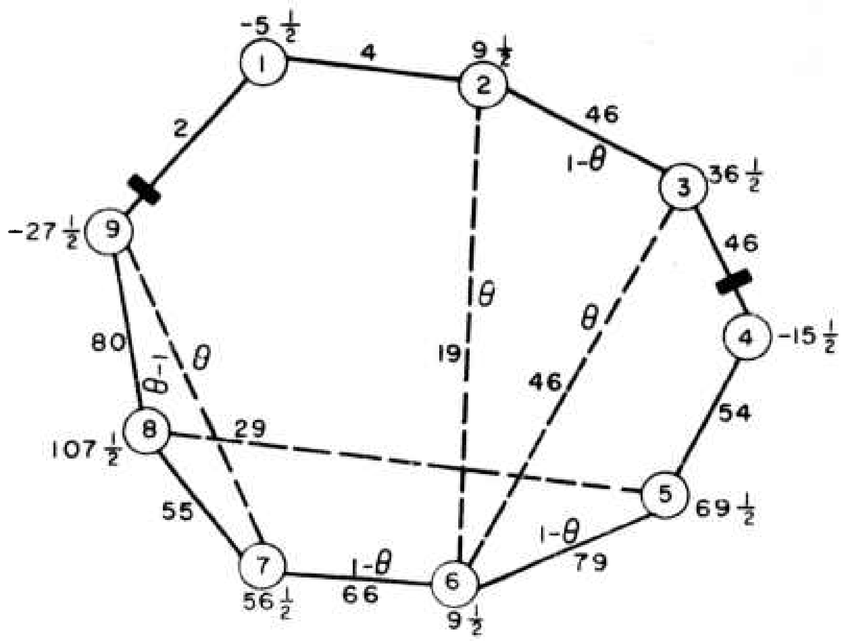

Figure 4

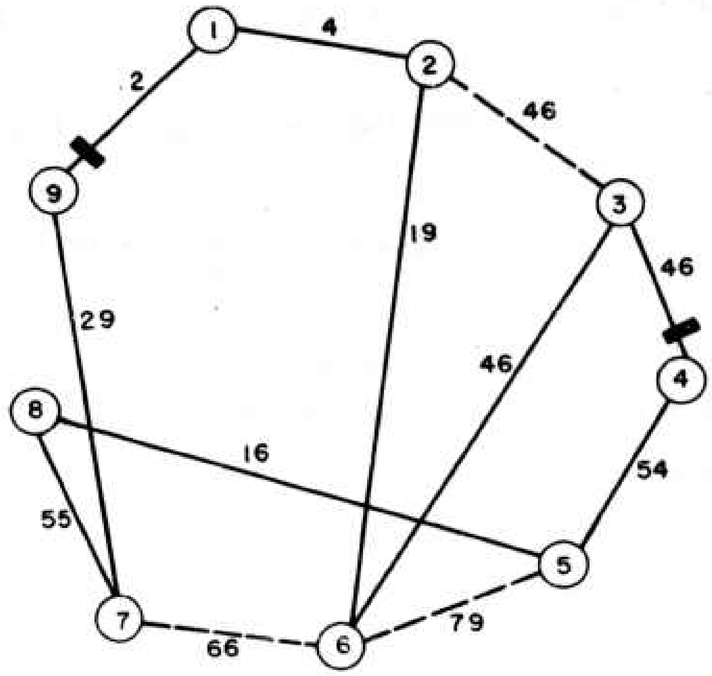

Figure 5

3. AN APPROXIMATION TECHNIQUE

The following approximation procedure has been suggested by A. W. Boldyreff.

Start with the first three cities. There is then only one tour of these, namely, 123. We now set

$$(4) \qquad F(i,j;4) = a_{i4} + a_{j4} - a_{ij} \qquad i,j = 1,2,3 .$$

Let k and ℓ be the cities such that

$$(5) \qquad F(k,\ell;4) = \min_{i,j=1,2,3} F(i,j;4) .$$

Then our tour of the first four cities replaces the link (k,ℓ) in the tour of the first three cities by the two links $(k,4)$ and $(4,\ell)$. We repeat this process, adding one city at a time, until we have a tour of all the cities.

To illustrate let us consider Table 1 and tabulate the procedure as follows.

					X		
1		2		3		1	city
	4		46		98		length of link
26		37		46		26	a_{14}

			X						
1		2		3		4		1	city
	4		46		46		26		length of link
84		50		95		54		84	a_{15}

					X						
1		2		5		3		4		1	city
	4		50		95		46		26		length of link
70		19		79		46		96		70	a_{16}

											X		
1		2		5		6		3		4		1	city
	4		50		79		46		46		26		length of link
19		77		88		66		75		26		19	a_{17}

					X										
1		2		5		6		3		4		7		1	city
	4		50		79		46		46		26		19		length of link
74		98		16		62		36		76		55		74	a_{18}

															X		
1		2		5		8		6		3		4		7		1	city
	4		50		16		62		46		46		26		19		length of link
2		69		57		80		90		82		61		29		2	a_{19}

1		2		5		8		6		3		4		7		9		1	final tour
	4		50		16		62		46		46		26		29		2		length of link

At each stage the X marks the link which is replaced. In this example the final tour has a length of 281 while the optimal tour has a length of 232. The error is 21%. In most

cases considered here the error is less than this, as can be seen from the comparison in the tables of Section 5.

There are many ways in which the accuracy of this technique may be improved, but it is felt that the resulting complexity would probably result in a relatively small increase in accuracy.

City	1	2	3	4	5	6	7	8	9
1									
2	4								
3	98	46							
4	26	37	46						
5	84	50	95	54					
6	70	19	46	96	79				
7	19	77	75	26	88	66			
8	74	98	36	76	16	62	55		
9	2	69	82	61	57	90	29	80	

Iteration	Tour	Length of Tour
0	1 2 3 4 5 6 7 8 9	432
1	1 2 6 3 4 5 8 7 9	271
2	1 2 6 3 8 5 4 7 9	232
Approximation Technique	1 2 5 8 6 3 4 7 9	281

Table 1

City	1	2	3	4	5	6	7	8	9
1									
2	96								
3	19	49							
4	50	58	22						
5	70	53	56	43					
6	59	89	50	82	62				
7	22	62	24	7	3	20			
8	40	35	75	72	46	24	69		
9	89	8	49	65	84	62	41	9	

City

Iteration	Tour	Length of Tour
0	1 2 3 4 5 6 7 8 9	459
1	1 2 3 4 5 6 7 9 8	382
2	1 3 4 5 2 6 7 9 8	336
3	1 3 4 7 6 5 2 9 8	240
4	1 3 4 7 6 8 9 2 5	232
5	1 3 4 5 2 9 8 6 7	220
6	1 3 4 7 5 2 9 8 6	204
Approximation Technique	1 4 7 5 2 9 8 6 1	225

Table 2

City	1	2	3	4	5	6	7	8	9
1									
2	12								
3	13	93							
4	85	1	86						
5	30	31	21	68					
6	10	19	1	98	98				
7	38	99	18	71	72	21			
8	22	25	8	30	87	28	70		
9	42	58	4	84	59	30	44	33	

City

Iteration	Tour	Length of Tour
0	1 2 3 4 5 6 7 8 9	523
1	1 2 4 3 5 6 7 8 9	384
2	1 2 4 8 9 5 3 7 6	205
3	1 2 4 8 9 6 7 3 5	196
4	1 2 4 8 9 7 6 3 5	193
5	1 2 4 8 6 7 9 3 5	191
6	1 6 7 3 9 8 4 2 5	178
Approximation Technique	1 8 2 4 5 3 9 7 6	216

Table 3

City	1	2	3	4	5	6	7	8	9
1									
2	95								
3	63	17							
4	23	69	60						
5	95	7	21	72					
6	55	49	20	86	75				
7	79	32	50	57	23	53			
8	42	80	99	61	29	16	78		
9	12	98	73	48	8	15	2	33	

Iteration	Tour	Length of Tour
0	1 2 3 4 5 6 7 8 9	495
1	1 4 5 2 3 7 6 8 9	283
2	1 4 3 2 5 7 6 8 9	244
3	1 4 8 6 3 2 5 7 9	185
Approximation Technique	1 4 7 9 5 2 3 6 8	192

Table 4

City	1	2	3	4	5	6	7	8	9
1									
2	63								
3	64	95							
4	24	18	76						
5	76	85	74	15					
6	38	65	68	76	11				
7	38	90	90	75	13	31			
8	51	43	77	92	26	6	51		
9	92	47	54	55	81	32	73	45	

Iteration	Tour	Length of Tour
0	1 2 3 4 5 6 7 8 9	479
1	1 2 9 8 7 6 5 4 3	403
2	1 2 4 5 6 7 8 9 3	352
3	1 2 4 5 7 6 8 9 3	309
4	1 3 9 2 4 5 8 6 7	299
5	1 3 9 6 8 2 4 5 7	283
Approximation Technique	1 8 6 7 5 4 2 9 3	299

Table 5

City	1	2	3	4	5	6	7	8	9
1									
2	47								
3	50	14							
4	50	33	66						
5	20	69	38	62					
6	8	26	86	28	82				
7	13	90	21	71	12	21			
8	35	55	99	2	65	52	27		
9	9	21	12	52	52	42	97	55	

Iteration	Tour	Length of Tour
0	1 2 3 4 5 6 7 8 9	383
1	1 5 6 7 8 4 3 2 9	262
2	1 5 4 8 7 6 2 3 9	193
3	1 5 7 8 4 6 2 3 9	150
Approximation Technique	1 6 4 8 2 9 3 7 5	179

Table 6

City	1	2	3	4	5	6	7	8	9
1									
2	42								
3	4	28							
4	40	8	1						
5	86	37	61	74					
6	49	88	94	20	64				
7	12	3	37	16	44	14			
8	31	41	69	13	65	89	92		
9	19	72	3	21	87	45	73	30	

City

Iteration	Tour	Length of Tour
0	1 2 3 4 5 6 7 8 9	364
1	1 3 4 8 2 5 7 6 9	218
2	1 3 9 6 7 5 2 4 8	199
3	1 3 9 8 4 2 5 6 7	185
Approximation Technique	1 7 6 5 2 8 4 3 9	204

Table 7

City	1	2	3	4	5	6	7	8	9
1									
2	15								
3	76	10							
4	8	81	22						
5	97	66	75	54					
6	88	21	93	77	1				
7	15	97	56	49	75	41			
8	22	48	1	34	43	58	36		
9	61	31	91	9	97	96	41	19	

Iteration	Tour	Length of Tour
0	1 2 3 4 5 6 7 8 9	259
1	1 7 9 8 3 4 5 6 2	189
2	1 7 9 4 3 8 5 6 2	168
Approximation Technique	1 2 3 8 9 7 6 5 4	190

Table 8

City	1	2	3	4	5	6	7	8	9
1									
2	69								
3	73	14							
4	55	38	16						
5	50	69	56	38					
6	53	73	86	51	25				
7	44	37	25	48	14	88			
8	41	99	9	75	57	53	13		
9	63	39	78	71	69	64	98	37	

Iteration	Tour	Length of Tour
0	1 2 3 4 5 6 7 8 9	363
1	1 7 8 9 2 3 4 5 6	279
2	1 4 3 2 9 8 7 5 6	266
Approximation Technique	1 9 8 7 2 3 4 5 6	296

Table 9

City	1	2	3	4	5	6	7	8	9
1									
2	92								
3	82	89							
4	54	59	49						
5	5	93	8	36					
6	36	99	58	93	15				
7	70	51	5	39	1	52			
8	16	37	76	6	32	90	99		
9	65	57	51	87	1	94	54	73	

Iteration	Tour	Length of Tour
0	1 2 3 4 5 6 7 8 9	570
1	1 2 3 7 6 5 4 8 9	479
2	1 9 5 6 7 3 2 4 8	308
3	1 6 7 3 2 9 5 4 8	298
4	1 6 5 9 2 7 3 4 8	236
Approximation Technique	1 6 2 8 4 7 3 9 5	279

Table 10

Distance Table same as in Table 10

Iteration	Tour	Length of Tour
0	1 7 8 6 9 4 5 2 3	740
1	1 6 8 4 9 5 2 3 7	477
2 and 3	1 6 8 4 3 7 2 9 5	300
4	1 6 5 9 2 7 3 4 8	236

Table 11

REFERENCES

[1] Dantzig, G., D. Fulkerson, and S. Johnson, "Solution of a Large Scale Traveling Salesman Problem," Jour. of the Oper. Res. Soc. of Amer., Vol. 2, No. 4, Nov., 1954, pp. 393-410.